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Crossing-free segments and triangles in point configurations

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Abstract

Consider the $\binom{n}{3}$ triangles determined by some n points in general position in 3-dimensional Euclidean space. We prove that if $n \geqslant 6$, then there are at most $3n-6$ such triangles that do not intersect any other triangle in the arrangement, and this result cannot be improved upon. We also study a few related problems in the plane and in higher dimensions.

Keywords: Point configurations; Halving planes; Planar graphs; Crossing simplices

1. Introduction

An *n-configuration* is a set of n points in d-space in general position, i.e. no $i+1$ points lie in a common $(i-1)$-flat, $1 \leqslant i \leqslant d$. A *segment* in an n-configuration is the convex hull of two of the points in the configuration, a *line* in an n-configuration is the affine hull of two points, and a *triangle* in an n-configuration is the convex hull of three points.

A configuration, together with a set of its segments is often referred to as a *geometric graph*. Geometric graphs have been studied extensively for the last two decades, see [11] for a survey.

In the plane we investigate the maximal possible number of *free segments*, i.e. segments which do not cross any other segment (we use the term cross for intersection of the relative interior). Here we sharpen previous results by Ringel [10], Harborth and Mengersen [7]. In particular, there is exactly one type of configuration which attains the (previously known) maximum number of $2n-2$. Otherwise, $2n-h$ is an upper bound,

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where h is the number of vertices of the convex hull of the configuration. *Absolutely free segments* are segments which are not crossed by any line in the configuration. We show that there cannot be more than n absolutely free segments.

In 3-space, a triangle in an n-configuration is *free*, if it is not crossed by any other triangle in the configuration. For $n \geqslant 6$, a tight upper bound of $3n-6$ for the number of free triangles is shown. This is in contrast to the situation for *segment free triangles*, i.e. triangles which are crossed by no segment in the configuration. The number of such segment free triangles can be as large as quadratic in n. Note that if two triangles cross, then at least one of the two must be crossed by a segment!

We also present some generalizations to d-space.

1.1. A motivating application

Let us briefly elaborate on an application of the type of bounds we consider in this paper. Consider an n-configuration in 3-space, n odd. A triangle in the configuration is called *halving*, if the plane spanned by this triangle bisects the remaining $n-3$ points into equal halves. Bounds on the maximum possible number of halving triangles have been extensively studied in [3,2,5,1] with a currently best upper bound of $O(n^{8/3})$ from [4]. Here we present a proof which is very similar to [4] but somewhat shorter. It is based on the fact that any line crosses at most $(n^2-1)/8$ halving triangles (see [4]), and its implication that the number c of crossings between halving triangles and segments in the configuration cannot be more than $\binom{n}{2} \cdot n^2/8 < n^4/16$.

So we are looking for a bound on t, the number of halving triangles, provided we have a bound on c, the number of crossings with segments. In Proposition 4.3 we will provide a simple argument for the case that the triangles are segment free: $c=0$ implies that $t \leqslant n^2 - n$, for all $n \geqslant 0$. Of course, this can be generalized to $t \leqslant n^2 - n + c$, since we can delete a triangle for each crossing arriving at a segment free collection of triangles after c deletions.

A better enhancement of the segment free bound to the situation of general c can be obtained by the following probabilistic argument (which can be considered as an application of Erdős' deletion trick). To this end assume that we have an n-configuration $\mathscr{V}$ with a collection $\mathscr{T}$ of t of its triangles, such that there are c crossings between triangles in $\mathscr{T}$ and segments in the configuration. Choose some parameter p, $0 < p \leqslant 1$, and take a sample of the points in $\mathscr{V}$ by selecting each point with probability p independently. This will give an N-configuration for some $N \leqslant n$. Let T be the number of triangles from $\mathscr{T}$ that survived in the sample (i.e., all three determining points were selected), and let C be the number of surviving crossings (i.e., both the triangle and the segment involved in the crossing survived). We know that $T \leqslant N^2 - N + C = 2\binom{N}{2} + C$, and so this relation must be true also for the expectations of these random variables, i.e.

$$E(T) \leqslant 2E\left(\binom{N}{2}\right) + E(C)$$

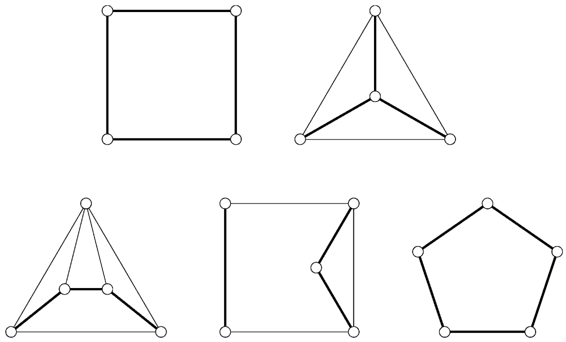

Fig. 1. All possible configurations on 4 resp. 5 points.

(because of linearity of expectations). The expectations involved in this inequality are easy to determine and yield the inequality

$$p^3 t \leqslant 2 \binom{n}{2} p^2 + p^5 c < p^2 n^2 + p^5 c,$$

or $t < p^{-1} n^2 + p^2 c$, for all $0 < p \leqslant 1$. Now set $p = n^{2/3}(2c)^{-1/3}$ to obtain a bound of

$$t < \frac{3\sqrt[3]{2}}{2} n^{4/3} c^{1/3},$$

for $c \geqslant n^2/2$.

For the problem of halving triangles we had derived $c < n^4/16$, which now yields an upper bound of

$$\tfrac{3}{4} n^{8/3}$$

on the number of halving triangles in an n-configuration in 3-space.

We have included this proof in order to show how the bounds of this paper can be generalized to the situation where there are few (rather than no) crossings present.

2. Straight line drawings of complete graphs

In this section we restrict ourselves to planar n-configurations, and we investigate free and absolutely free segments of such configurations.

For example, the sides of the convex hull of a configuration are always free. If the configuration is the vertex set of a convex polygon, then the free segments are exactly the edges of the polygon, and the same is true for the absolutely free segments. The next picture (Fig. 1) shows a complete list of possible types of 4- and 5-configurations, together with their free segments. Absolutely free segments are bold.

Let $f(n)$ and $a(n)$ denote the maximum number of free and absolutely free segments, respectively, an n-configuration may have. Note that the geometric graph formed by

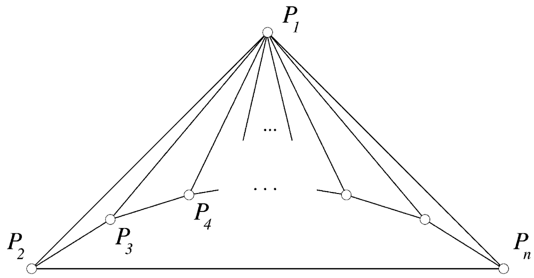

Fig. 2. A configuration of type $E(n)$.

the free segments is planar, hence, for $n \geqslant 3$,

$$n \leqslant a(n) \leqslant f(n) \leqslant 3n-6.$$

We call the configuration $\{P_1, P_2, \ldots, P_n\}\,(n \geqslant 4)$ of type $E(n)$ if $P_2P_3\ldots P_n$ is a convex polygon contained in triangle $P_1P_2P_n$. Note that, if $n \geqslant 5$, then any configuration of type $E(n)$ uniquely determines the vertex P_1 and the segment P_2P_n which we call, respectively, the *apex* and the *base* of the configuration. In such a configuration the free segments are P_1P_i for $2 \leqslant i \leqslant n$, P_iP_{i+1} for $2 \leqslant i \leqslant n-1$ and P_2P_n, see Fig. 2. Therefore $f(n) \geqslant 2n-2$ for $n \geqslant 4$, see also [8].

The relation $f(n) \leqslant 2n-2$ was first observed by Ringel [10]. In fact, he proved that in any drawing of the complete graph K_n in the plane with Jordan arcs there are at most $2n-2$ crossing free edges. See also [7] for a brilliant short proof. Moreover, it is also inherent in either of these proofs that the only n-configurations with $2n-2$ free segments are those of type $E(n), n \geqslant 4$. Here we first give an alternative proof which yields a stronger statement.

Theorem 2.1. *Suppose that $\mathscr{V}$ is a configuration of $n \geqslant 3$ points which is not of type $E(n)$. Then $\mathscr{V}$ has at most $2n-h$ free segments, where h denotes the number of vertices of the convex hull of $\mathscr{V}$. Moreover, this result cannot be improved upon.*

Proof. Fix a configuration $\mathscr{U} = \{P_1, \ldots, P_n\}$ of $n \geqslant 4$ points. For convenience, let $f(\mathscr{U})$ denote the number of free segments in $\mathscr{U}$. Choose an arbitrary point $P \notin \mathscr{U}$ inside $\mathrm{conv}(\mathscr{U})$ which does not violate the general position condition. We want to show $f(\mathscr{U} \cup \{P\}) \leqslant f(\mathscr{U}) + 2$, which will then be used to prove the first statement of the theorem inductively. Consider an arbitrary triangulation of $conv(\mathscr{U})$ and let $\Delta = QRS$ be the triangle that contains P.

Suppose first that each side of Δ is a free segment in $\mathscr{U}$. Let $T \in \mathscr{U} \setminus \{Q, R, S\}$. Since T has to lie outside Δ, PT intersects the boundary $\mathrm{bd}(\Delta)$ of Δ. Therefore it intersects one of its sides, which cannot be a free segment in $\mathscr{U} \cup \{P\}$. On the other hand, among the segments PP_i, only PQ, PR and PS may be free segments in $\mathscr{U} \cup \{P\}$. Thus, $f(\mathscr{U} \cup \{P\}) \leqslant (f(\mathscr{U}) - 1) + 3 = f(\mathscr{U}) + 2$.

Suppose next that QR, RS, QS are not all free segments in $\mathscr{U}$. Then there are two points $P_i, P_j \in \mathscr{U}$ such that P_i, P_j intersects $\mathrm{bd}(\Delta)$. Since Δ contains no point of $\mathscr{U}$ in its interior, $|\mathrm{bd}(\Delta)\cap P_iP_j|=2$. Therefore, if we restrict the topology of the plane to Δ, then $s=\Delta\cap P_iP_j$ separates two vertices of Δ. However, $t=PQ\cup PR\cup PS$ is a connected set which contains the separated vertices, therefore $s\cap t\neq\emptyset$. Thus, s, and also P_iP_j, intersects one of the segments PQ, PR, PS, which then cannot be a free segment in $\mathscr{U}\cup\{P\}$. Consequently, we have $f(\mathscr{U}\cup\{P\})\leqslant f(\mathscr{U})+2$ in this case, too.

For the induction, we can use the above argument to trace the statement back to a smaller case if there is a configuration $\mathscr{V}'\subset\mathscr{V}$ of $n-1$ points such that $\mathrm{conv}(\mathscr{V})$ and $\mathrm{conv}(\mathscr{V}')$ have the same vertex set and $\mathscr{V}'$ is not of type $E(n-1)$. This is the case if either $n>h\geqslant 4$ or $h=3$, $n\geqslant 7$. (For the latter case, assume that every such sub-configuration $\mathscr{V}'$ is of type $E(n-1)$. If $\mathscr{V}_1'$ and $\mathscr{V}_2'$ are two such subsets of $\mathscr{V}$, then $\mathscr{V}_1'\cap\mathscr{V}_2'$ is of type $E(n-2)$, $n-2\geqslant 5$. Thus, its apex P and its base are uniquely determined. Consequently, every $\mathscr{V}'$ has the same apex P, and every set $\mathscr{V}'\setminus\{P\}$, and thus also $\mathscr{V}\setminus\{P\}$ is in convex position. It follows that $\mathscr{V}$ is of type $E(n)$, a contradiction.) Thus, to complete the proof of the first part of the theorem one only has to check that it is valid in the initial cases $n=h\geqslant 4$ and $h=3$, $n=6$. (If $h=3$, $n\leqslant 5$, then $\mathscr{V}$ is of type $E(n)$.) We leave the details to the reader.

To see that the result is sharp, let first $n\geqslant 6$, and consider a configuration $\{P_1,\ldots,P_{n-1}\}$ of type $E(n-1)$ coupled with a point P_n inside the convex polygon $P_2P_3\ldots P_{n-1}$. P_1P_n intersects exactly one of the segments P_jP_{j+1} $(2\leqslant j\leqslant n-2)$, say P_iP_{i+1}. In the configuration $\mathscr{V}=\{P_1,P_2,\ldots,P_n\}$, whose convex hull is triangle $P_1P_2P_{n-1}$, there are exactly $2n-3$ free segments, namely P_1P_j $(2\leqslant j\leqslant n-1)$, P_jP_{j+1} $(2\leqslant j\leqslant n-2, j\neq i)$, P_2P_{n-1}, P_2P_n and $P_{n-1}P_n$. This proves the assertion for $h=3$.

To settle the general case, let $n\geqslant h\geqslant 4$. Let first $\{P_1,\ldots,P_{n-h+3}\}$ be any configuration of type $E(n-h+3)$. Next, choose an arbitrary $2\leqslant i\leqslant n-h+2$ and points $Q_1,\ldots,Q_{h-3}$ that lie in the angular region $P_iP_1P_{i+1}$ such that $P_1P_2Q_1\ldots Q_{h-3}P_{n-h+3}$ is a convex h-gon. In the configuration $\mathscr{V}=\{P_1,\ldots,P_{n-h+3},Q_1,\ldots,Q_{h-3}\}$ there are exactly $2n-h$ free segments, namely P_1P_j $(2\leqslant j\leqslant n-h+3)$, P_jP_{j+1} $(2\leqslant j\leqslant n-h+2, j\neq i)$, Q_jQ_{j+1} $(1\leqslant j\leqslant h-4), P_2Q_1$ and $P_{n-h+3}Q_{h-3}$. This proves the assertion for an arbitrary $h\geqslant 4$. $\square$.

On the other hand, it is not difficult to see that any n-configuration contains at least five free segments if $n\geqslant 5$. It is obvious if the convex hull of the points has at least 5 vertices. In fact, if the convex hull is a quadrilateral ($n\geqslant 5$), or a triangle ($n\geqslant 4$), then there are at least six free edges in the configuration. The easy proof is left to the reader. As a counterpart, the following result is known.

Theorem 2.2 (Harborth and Thürmann [8]). *If $n\geqslant 8$, then there is an n-configuration that contains exactly five free segments.*

There are similar phenomena concerning absolutely free segments.

Theorem 2.3. *In a planar configuration of $n\geqslant 3$ points there are at most n absolutely free segments. Moreover, there are n absolutely free segments if and only if the points are in convex position.*

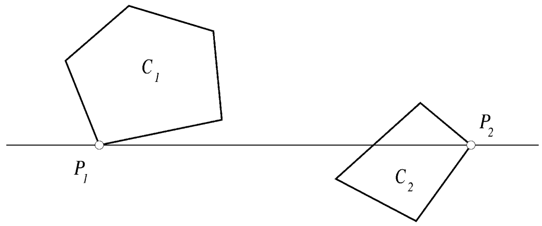

Fig. 3.

Proof. Let $\mathscr{V}$ be any configuration of n points and consider the graph $\mathscr{G}=(\mathscr{V},\mathscr{E})$ of absolutely free segments. Let V be any vertex of $\mathscr{G}$. Had V four or more neighbours in $\mathscr{G}$, at least three of them would lie in the same half-plane supported by a line through V. This is easily seen to be impossible. Thus, the maximum degree of $\mathscr{G}$ is at most 3.

Claim 2.4. *Suppose that $PQ, RS \in \mathscr{E}$ are such that vertices Q and S lie in the same* (*open*) *half-plane H supported by PR. Then neither P nor R has any other neighbour in H.*

Returning to the proof of Theorem 2.3, note that it follows from Claim 2.4 that if $\mathscr{G}$ has a vertex of degree $\geqslant 2$, then no other vertex has degree $\geqslant 3$. Therefore the sum of the degrees of the vertices of $\mathscr{G}$,

$$2|\mathscr{E}| = \sum_{P\in\mathscr{V}} deg(P) \leqslant \max\{3+(n-1)\cdot 1, n\cdot 2\} = 2n,$$

proving the first part of the assertion.

To find the extremal configurations, notice that if $|\mathscr{E}|=n$, then each vertex has degree 2. Thus, $\mathscr{G}$ is a disjoint union of cycles. These cycles are closed polygons, which must be convex for obvious reasons. Suppose there are two disjoint cycles $\mathscr{C}_1$ and $\mathscr{C}_2$. Then there exist vertices P_i of $\mathscr{C}_i$ such that one of the half-planes supported by P_1P_2 fully contains $\mathscr{C}_1$ but contains exactly one vertex of $\mathscr{C}_2$ different from P_2, see Fig. 3. This contradicts Claim 2.4. Thus, $\mathscr{G}$ is a closed convex n-gon. $\square$.

Theorem 2.5. *If $n \geqslant 6$, then there is an n-configuration that contains no absolutely free segments.*

Remark. This is not valid for $2 \leqslant n \leqslant 5$, see Fig. 1.

Before the proof we do a little preparation. Consider a regular polygon with an odd number of vertices. The diagonals of the polygon divide it into a finite number of regions. The region containing the centre of the polygon will be referred to as *the central region*.

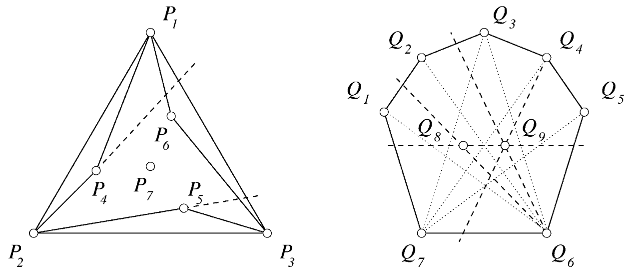

Fig. 4. Small configurations without absolutely free segments.

Proposition 2.6. *Let $\mathscr{P} = P_1P_2\ldots P_k$ be a regular k-gon ($k \geqslant 5$ odd), and P any point in its central region. Let $\mathscr{Q}$ be any configuration that contains $P, P_1, P_2, \ldots, P_k$. Then*

(i) *PP_i ($1 \leqslant i \leqslant k$) is not free in $\mathscr{Q}$,*
(ii) *no diagonal of $\mathscr{P}$ is free in $\mathscr{Q}$, and*
(iii) *no side of $\mathscr{P}$ is absolutely free in $\mathscr{Q}$.*

Proof. To verify the first two assertions one only need to find a diagonal of $\mathscr{P}$ that intersects the segment in question. To prove (iii), consider any side of $\mathscr{P}$. The line connecting the opposite vertex of $\mathscr{P}$ to P then intersects the given side. □

Proof of Theorem 2.5. Assume first that $n \geqslant 6$ is even. Let $P_1P_2\ldots P_{n-1}$ be a regular $(n-1)$-gon with P_n in its central region. It follows immediately from the proposition that there is no absolutely free segment in the configuration $\mathscr{V} = \{P_1, P_2, \ldots, P_n\}$.

Next, let $n \geqslant 11$ odd. Set $k = l = (n-1)/2$ if $n \equiv 3 \pmod 4$ and let $k = l + 2 = (n+1)/2$ otherwise. Let $\mathscr{P} = P_1P_2\ldots P_k$ and $\mathscr{Q} = Q_1Q_2\ldots Q_l$ be regular k- resp. l-gons with common centre P such that $\mathscr{Q}$ lies in the central region of $\mathscr{P}$ and $\mathscr{V} = \mathscr{P} \cup \mathscr{Q} \cup \{P\}$ is in general position. Based on the first part of the proof, it is enough to see that P_iQ_j ($1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$) is not absolutely free. However, this follows from part (i) of the proposition.

Appropriate configurations on 7 and 9 points, respectively, are depicted on Fig. 4. Here $P_1P_2P_3$ and $P_4P_5P_6$ are both equilateral triangles centered at P_7, the latter obtained from the triangle formed by the midpoints of the sides of $P_1P_2P_3$ when a small rotation about P_7 is applied. Points $Q_1, Q_2, \ldots, Q_7$ are appropriate vertices of a regular 10-gon, and Q_8Q_9 is a small segment parallel to Q_1Q_5, which contains the centre of the polygon. □

3. Crossing-free triangles in 3-space

An *i-simplex* in a configuration in d-space is any i-simplex which is the convex hull of some $i+1$ points in the configuration. We use *simplex* for a d-simplex and *hyper-triangle* for a $(d-1)$-simplex. A hyper-triangle is said to be free if it does not intersect any other hyper-triangle in the configuration in its relative interior. In a configuration $\mathscr{V}$, denote by vert($\mathscr{V}$) the vertex set of the convex hull of $\mathscr{V}$, and let $f_d(\mathscr{V})$ denote the number of free hyper-triangles in $\mathscr{V}$. Let, for $n > d, f_d(n)$ denote the maximum number of free hyper-triangles a d-dimensional n-configuration may have. Thus, by Ringel's Theorem, $f_2(n) = f(n) = 2n - 2$ for $n \geqslant 4$. The aim of this section is to prove a similar result for free triangles in 3-space, namely $f_3(n) = 3n - 6$.

Theorem 3.1. *In any configuration of $n \geqslant 6$ points in* 3*-space, there are at most* $3n-6$ *free triangles, and this result is best possible. Moreover, in any extremal configuration* $\mathscr{V}$, conv($\mathscr{V}$) *is a tetrahedron.*

The proof is by induction, and is similar to that of Theorem 2.1. Its essence is the following lemma. For further convenience, we state it in a more general form than it is needed here.

Lemma 3.2. *Let $\mathscr{V}$ be any configuration of $n \geqslant d + 2$ points in d-space, and $P \notin \mathscr{V}$ an interior point of* conv($\mathscr{V}$) *such that $\mathscr{V} \cup \{P\}$ is in general position. Then* $f_d\ (\mathscr{V} \cup \{P\}) \leqslant f_d(\mathscr{V}) + d$.

Proof. A *triangulation* of $\mathscr{V}$ is a (geometric) simplicial complex whose vertex set is $\mathscr{V}$ and whose body is conv($\mathscr{V}$). In a triangulation T of $\mathscr{V}$, let Δ_T denote the d-simplex which contains P. Since $\mathscr{V} \cup \{P\}$ is in general position, $P \in \mathrm{int}(\Delta_T)$. The interior of Δ_T can be expressed, in a unique way, as the intersection of $d + 1$ open half-spaces $H_1, \ldots, H_{d+1}$. For any point $Q \in \mathscr{V}$, let us introduce the type of Q as the number of such open half-spaces H_i which contain Q. Thus, the type of Q is an integer between 1 and d, the vertices of Δ_T being of type 1.

Let Γ be any hyper-triangle in $\mathscr{V}$. If every vertex of Γ is of type 1, then Γ does not cross the boundary of Δ_T. Indeed, for each vertex of Γ there is only one half-space H_j whose complement $\bar{H}_j$ does not contain the given vertex. Since there are more such half-spaces than vertices of Γ, there is an $1 \leqslant i \leqslant d + 1$ such that every vertex of Γ is contained in $\bar{H}_i$, hence $\Gamma \subset \bar{H}_i$. On the other hand, $\mathrm{int}(\Delta_T) \subset H_i$. Thus, Γ cannot cross bd(Δ_T) and we obtain the following

Proposition 3.3. *Suppose that T is a triangulation of $\mathscr{V}$ such that every point of $\mathscr{V}$ has type* 1*. Then each facet of Δ_T is a free hyper-triangle in $\mathscr{V}$.*

Now we fix a triangulation T of $\mathscr{V}$ and suppose first that every point of $\mathscr{V}$ is of type 1. Choose any $Q \in \mathscr{V}$ which is not a vertex of Δ_T. Then we can enumerate the vertices of Δ_T as $P_1, P_2, \ldots, P_d, P_{d+1}$ such that P_{d+1} is in the simplex $P_1 \ldots P_d Q$ and in the simplex $P_1 \ldots P_{d-1} PQ$, while P_d is not in the latter simplex. It follows that the edge $P_d P_{d+1}$ crosses a hyper-triangle of the simplex $P_1 \ldots P_{d-1} PQ$. Thus, no hyper-triangle incident to $P_d P_{d+1}$ can be free.

This means, that among the $\binom{d+2}{d}$ hyper-triangles determined by points $P_1,\dots,P_{d+1}$ and P, at most $\binom{d+2}{d}-\binom{d}{d-2}=2d+1$ can be free in $\mathscr{V}\cup\{P\}$. On the other hand, the $d+1$ hyper-triangles determined by $P_1,\dots,P_{d+1}$ are free in $\mathscr{V}$, as it is guaranteed by the Proposition. Thus,

$$f_d(\mathscr{V}\cup\{P\})\leqslant f_d(\mathscr{V})-(d+1)+(2d+1)=f_d(\mathscr{V})+d,$$

what was to be proved.

For the remaining cases, suppose that there is a point $Q\in\mathscr{V}$ of type i for some $2\leqslant i\leqslant d$. The hyperplanes spanned by the facets of $\Delta_T=P_1P_2\dots P_{d+1}$, respectively, divide the space into $2^{d+1}-1$ (open) regions. The region which contains Q have exactly i vertices, and we may assume that these vertices are $P_1,\dots,P_i$. The $(d-i+1)$-simplex $P_{i+1}\dots P_{d+1}Q$ crosses the $(i-1)$-simplex $P_1\dots P_i$ at a point Q', and if we connect the $(d-i+1)$-simplex $P_{i+1}\dots P_{d+1}Q'$ to each $(i-2)$-face of $P_1\dots P_i$, we obtain a subdivision of Δ_T into i simplices. One of these, say $P_2\dots P_{d+1}Q'$, has P in its interior. Thus, segment PP_1 intersects one of its facets. Without loss of generality, we may assume that this facet is $P_3\dots P_{d+1}Q'$, in which case PP_1 crosses the hyper-triangle $P_3\dots P_{d+1}Q$ of configuration $\mathscr{V}\cup\{P\}$. Therefore, no hyper-triangle incident to PP_1 can be a free hyper-triangle in $\mathscr{V}\cup\{P\}$. This implies that if Γ is a free hyper-triangle in $\mathscr{V}\cup\{P\}$ but is not free in $\mathscr{V}$, then one vertex of Γ is P and its $d-1$ other vertices are among $P_2,\dots P_{d+1}$, meaning that there cannot be more than d such hyper-triangles at all. This completes the proof of the lemma. □

Proof of Theorem 3.1. Let $\mathscr{V}$ be an arbitrary n-configuration in 3-space. We will distinguish between two cases. Suppose first that $h=|\mathrm{vert}(\mathscr{V})|\geqslant 5$. Define $\mathscr{W}=\mathrm{vert}(\mathscr{V})$ and $\mathscr{P}=\mathrm{conv}(\mathscr{W})$. It is not difficult to see, and in fact we will prove it in a more general form (see Proposition 4.2), that Γ is a free triangle in $\mathscr{W}$ if and only if Γ is a face of $\mathscr{P}$. Note that $\mathscr{P}$ has triangular faces, and hence Euler's relation yield $f(\mathscr{W})=2|\mathscr{W}|-4$. In this case, Lemma 3.2 can be applied $n-h$ times to obtain that $f(\mathscr{V})\leqslant f(\mathscr{W})+3(n-h)=3n-h-4<3n-6$.

Next, suppose that $|\mathrm{vert}(\mathscr{V})|=4$. It is easy to check that $f(\mathscr{V})=4$ if $n=4$, $f(\mathscr{V})=10$ if $n=5$, and $f(\mathscr{V})=12$ if $n=6$. If $n\geqslant 6$, then we may choose a configuration of 6 points $\mathscr{W}$ such that $\mathrm{vert}(\mathscr{V})\subset\mathscr{W}\subseteq\mathscr{V}$. Applying Lemma 3.2 $n-6$ times we get $f(\mathscr{V})\leqslant f(\mathscr{W})+3(n-6)=3n-6$.

To complete the proof of the theorem we only have to construct, for each $n\geqslant 6$, an n-configuration $\mathscr{V}=\mathscr{V}_n$ with $3n-6$ free triangles. Consider first an auxiliary (planar) configuration $\mathscr{U}=\{P_1,\dots,P_{n-1}\}$ of type $E(n-1)$. Choose a sphere of very large radius passing through points P_1,P_2 and P_{n-1}, and lift the points of $\mathscr{U}$ up to the sphere, orthogonally to the plane. This way we obtain a configuration $\mathscr{U}'=\{P_1',\dots,P_{n-1}'\}$ in 3-space such that $P_1'=P_1$, $P_2'=P_2$, $P_{n-1}'=P_{n-1}$, and P_i' is very close to P_i for $3\leqslant i\leqslant n-2$. Furthermore, $P_1',\dots,P_{n-1}'$ are the vertices of a convex polytope. In particular, triangles $P_1'P_i'P_{i+1}'$ $(2\leqslant i\leqslant n-2)$ and $P_1'P_2'P_{n-1}'$ are free in $\mathscr{U}'$.

Choose now a point P_n very high above the triangle $P_1P_2P_{n-1}$, and consider the configuration $\mathscr{V}_n=\mathscr{U}'\cup\{P_n\}$. If P_n is high enough, then triangle $P_nX'Y'$ intersects triangle $P_nZ'V'$ if and only if segment XY intersects segment ZV in $\mathscr{U}$. Furthermore, triangle $P_nX'Y'$ can intersect triangle $Z'V'W'$ only if segment XY intersects triangle ZVW. Thus, the above mentioned $n-2$ triangles are free not only in $\mathscr{U}'$, but also in $\mathscr{V}_n$, and so are all triangles $P_nX'Y'$, whenever XY is a free segment in $\mathscr{U}$. Being $\mathscr{U}$ of type $E(n-1)$, this gives altogether $(n-2)+2(n-1)-2=3n-6$ different free triangles in $\mathscr{V}_n$. $\square$

4. Related problems and generalizations

There is a somewhat different phenomenon in higher dimensions. Namely, $f_d(n)$ is no more a linear function in n, and, if n is large enough, then in the extremal configurations the points are in convex position. Obviously, $f_d(d+1)=d+1$ and $f_d(d+2)=\binom{d+2}{d}$. The following result determines $f_d(n)$ for $n \geqslant d+3$ if $d \geqslant 8$.

Theorem 4.1. *For every $d \geqslant 4$ there is a positive integer $n_0(d)$ such that*

$$f_d(n)=\binom{n-\lfloor (d+1)/2\rfloor}{n-d}+\binom{n-\lfloor (d+2)/2\rfloor}{n-d}$$

$=\Theta(n^{\lfloor d/2\rfloor})$ *if $n \geqslant n_0(d)$. Moreover, in an n-configuration $\mathscr{V}$, $n \geqslant n_0(d)$, there are $f_d(n)$ free triangles if and only if $\mathscr{V}$ is the vertex set of a convex simplicial neighbourly polytope. In particular, $n_0(d)=d+3$ if $d \geqslant 8$.*

Proof. Let first $\mathscr{V}$ be any configuration of $n \geqslant d+2$ points such that $\operatorname{conv}(\mathscr{V})$ is a d-simplex. Choose a configuration $\mathscr{W}$ of $d+2$ points such that $\operatorname{vert}(\mathscr{V}) \subset \mathscr{W} \subseteq \mathscr{V}$. It follows from Lemma 3.2 that

$$f(\mathscr{V}) \leqslant f(\mathscr{W})+d(n-d-2)=dn-\frac{1}{2}(d+2)(d-1).$$

Suppose next that $\mathscr{V}$ is any configuration of $h \geqslant d+2$ points which are the vertices of a convex polytope $\mathscr{P}$, that is, $\mathscr{V}$ is in convex position. Obviously, every facet of $\mathscr{P}$ is a free hyper-triangle in $\mathscr{V}$. We can show that the converse is true.

Proposition 4.2. *Let $\mathscr{V}$ be in convex position. Then Γ is a free hyper-triangle in $\mathscr{V}$ if and only if Γ is a facet of* $\operatorname{conv}(\mathscr{V})$.

Proof. Suppose, for contradiction, that $\Gamma=P_1\ldots P_d$ is not a facet of $\mathscr{P}=\operatorname{conv}(\mathscr{V})$. Then there are two vertices R,Q of $\mathscr{P}$ which are separated by the hyperplane H that contains Γ. Segment RQ intersects H at a point P. We may assume that, inside the $(d-1)$-flat H, P lies in the half-space supported by $P_1\ldots P_{d-1}$ and containing Γ.

If P is on the boundary of Γ, then $\mathscr{V}$ is not in general position. If the $(d-2)$-simplex $P_1\ldots P_{d-1}$ has a facet F such that $\operatorname{conv}(F\cup\{P\})$ crosses Γ, then hyper-triangle $\operatorname{conv}(F\cup\{Q,R\})$, too, crosses Γ, and hence Γ is not free. Finally, if P_d is inside $P_1\ldots P_{d-1}P$, then $P_d\in\operatorname{conv}(P_1,\ldots,P_{d-1},Q,R)$, contradicting the assumption that the vertices of $\mathscr{P}$ are in convex position. $\square$

In the view of this proposition, it follows from the Upper Bound Theorem of McMullen [9] that $f_d(\mathscr{V}) \leqslant c(h,d)$, where

$$c(h,d) = \binom{h - \lfloor (d+1)/2 \rfloor}{h-d} + \binom{h - \lfloor (d+2)/2 \rfloor}{h-d}$$

is the number of facets a cyclic d-polytope with h vertices has, see [6]. Furthermore, $f_d(\mathscr{V}) = c(n,d)$ if and only if $\mathscr{P}$ is a simplicial neighbourly polytope.

Consider now any configuration $\mathscr{V}$ of $n \geqslant d+2$ points such that $h = |\mathrm{vert}(\mathscr{V})| \geqslant d+2$. It follows from Lemma 3.2 that

$$f(\mathscr{V}) \leqslant f(\mathrm{vert}(\mathscr{V})) + d(n-h) \leqslant c(h,d) + d(n-h) \leqslant c(n,d).$$

Moreover, if equality is attained in the last inequality, then $h = n$.

To complete the proof of the theorem we only have to note that if n is large enough, then $dn - \frac{1}{2}(d+2)\,(d-1) < c(n,d)$. This is the case, in particular, if $n \geqslant d+3$, assuming that $d \geqslant 8$. $\square$

We conclude this paper with raising a more general question. In a d-dimensional configuration, an i-simplex ($i < d$) is said to be j-free ($j < d$) if it does not intersect any j-simplex of the configuration. In a configuration $\mathscr{V}$, let $f^{i,j}(\mathscr{V})$ denote the number of j-free i-simplices, and let, for $n > d$, $f_d^{i,j}(n)$ denote the maximum number of j-free i-simplices a d-dimensional n-configuration may have. Thus, a free hyper-triangle in a d-dimensional configuration is a $(d-1)$-free $(d-1)$-simplex, and $f_d^{d-1,d-1}(n) = f_d(n)$.

Obviously, $f_d^{i,j}(n) = \binom{n}{i+1} = \Theta(n^{i+1})$ if $i+j < d$. It can be proved similarly to Theorems 3.1 and 4.1 that $f_3^{1,2}(n) = 3n-5$ if $n \geqslant 5$ and $f_d^{1,d-1} = \binom{n}{2}$ if $d \geqslant 4$. Moreover, a trivial lower bound $f_d^{i,j}(n) = \Omega(n^{\min\{i+1,\lfloor d/2 \rfloor\}})$ can be obtained by counting i-faces of cyclic polytopes. Let us conclude with a result which beats this lower bound.

Proposition 4.3. *For* $n \geqslant 4$, $\lfloor n/2 \rfloor (n-2) \leqslant f_3^{2,1}(n) \leqslant n^2 - 3n$.

Proof. To see the upper bound first, let $\mathscr{V}$ be any n-configuration in 3-space, $n \geqslant 4$, and $P \in \mathscr{V}$. Let $\mathscr{T}$ be the set of 1-free triangles in $\mathscr{V}$. Consider a small sphere centered at P. Each triangle in $\mathscr{T}$ incident to P intersects the sphere along an arc. Moreover, these arcs form a planar graph on $\leqslant n-1$ nodes. Thus, any $P \in \mathscr{V}$ is incident to at most $3n-9$ 1-free triangles, yielding $f_3^{2,1}(\mathscr{V}) \leqslant n^2 - 3n$.

To prove the lower bound, consider points P_i ($1 \leqslant i \leqslant \lceil n/2 \rceil$) in general position in 3-space, and points Q_j ($1 \leqslant j \leqslant \lfloor n/2 \rfloor$) such that $\mathscr{V} = \{P_1, \ldots, P_{\lceil n/2 \rceil}, Q_1, \ldots, Q_{\lfloor n/2 \rfloor}\}$ is in general position, and Q_j is very close to P_j for $1 \leqslant j \leqslant \lfloor n/2 \rfloor$. If the distances between P_j and Q_j ($1 \leqslant j \leqslant \lfloor n/2 \rfloor$) are small enough, then all triangles of the form $P_jQ_jR(1 \leqslant j \leqslant \lfloor n/2 \rfloor, R \in \mathscr{V} \setminus \{P_j, Q_j\})$ are 1-free in $\mathscr{V}$, hence the lower bound. $\square$

Problem 4.4.

(i) *Determine* $f_3^{2,1}(n)$.

(ii) *Find, in general, the order of magnitude of* $f_d^{i,j}(n)$.

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References

[1] P.K. Agarwal, B. Aronov, T.M. Chan, M. Sharir, On levels in arrangements of lines, segments, planes, and triangles, Discrete Comput. Geom. 19 (1998) 315–331.
[2] B. Aronov, B. Chazelle, H. Edelsbrunner, L. Guibas, M. Sharir, R. Wenger, Points and triangles in the plane and halving planes in space, Discrete Comput. Geom. 6 (1991) 435–442.
[3] I. Bárány, Z. Füredi, L. Lovász, On the number of halving planes, Combinatorica 10 (1990) 175–183.
[4] T.K. Dey, H. Edelsbrunner, Counting triangle crossings and halving planes, Discrete Comput. Geom. 12 (1994) 281–289.
[5] D. Eppstein, Improved bounds for intersecting triangles and halving planes, J. Combin. Theory Ser. A 62 (1993) 176–182.
[6] B. Grünbaum, Convex Polytopes, Wiley Interscience, London, 1967.
[7] H. Harborth, I. Mengersen, Edges with at most one crossing in drawings of the complete graph, in: R. Bodendiek, R. Henn (Eds.), Topics in Combinatorics and Graph Theory (Oberwolfach, 1990), Physica, Heidelberg, 1990, pp. 757–763.
[8] H. Harborth, Ch. Thürmann, Number of edges without crossings in rectilinear drawings of the complete graph, Congr. Numerantium 119 (1996) 79–83.
[9] P. McMullen, The maximum numbers of faces of a convex polytope, Mathematika 17 (1970) 179–184.
[10] G. Ringel, Extremal problems in the theory of graphs, in: M. Fiedler (Ed.), Theory of Graphs and its Applications (Smolenice, 1963), Publ. House Czechoslovak Acad. Sci., Prague, 1964, pp. 85–90.
[11] J. Pach, P.K. Agarwal, Combinatorial Geometry, Wiley-Interscience, New York, 1995.